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Common Fixed Point Theorems for Weak Contraction Mapping of Integral Type in Modular Spaces

R. A. Rashwan[1,*], **H. A. Hammad**[2]

[1]Department of Mathematics, Faculty of Science, Assuit University, Assuit 71516, Egypt

[2]Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt

*Corresponding Author: rr_rashwan54@yahoo.com

Abstract In this paper, we prove three common fixed point theorems for weak contraction mappings of integral type in modular spaces. In the first theorem we prove a common fixed point of $\rho-$compatible mappings satisfying a $(\phi-\Psi)-$weak contraction. The second theorem is another version of the first theorem. In the third theorem we study a common fixed point of $\rho-$compatible mappings in modular spaces involving altering distances of integral type. The results extended several similar results in metric and Banach spaces.

Keywords Common Fixed Point, Modular Spaces, $\rho-$Compatible Maps, Comparsion Function, Lebesgue-stieltjes Integrable Mapping, Altering Distance Function

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1 Introduction

Let (X,d) be a complete metric space and $f: X \to X$ a self mapping of X. Suppose that $F_f = \{x \in X : F(x) = x\}$ is the set of fixed points of f. The classical Banach's fixed point theorem is established in Banach [5] by using the following contractive defination: there exists $k \in [0,1)$ such that $\forall x, y \in X$, we have

$$d(fx, fy) \leq kd(x,y). \tag{1.1}$$

Rhoades [19] proved the following theorem:

Theorem 1.1 Let T be a mapping from a complete metric space (X,d) into itself satisfying

$$d(Tx,Ty) \leq d(x,y) - \phi(d(x,y)) \ \forall x,y \in X, \tag{1.2}$$

where $\phi : R^+ \to R^+$ is continuous and nondecreasing such that ϕ is positive on $R^+\backslash\{0\}$, $\phi(0) = 0$ and $\lim\limits_{t\to\infty} \phi(t) = \infty$. Then T has a unique fixed point in X.

We note that (1.1) is a special case of (1.2) by taking $\phi(t) = (1-k)t$ for $0 \leq k < 1$.

Branciari [8] and Rhoades [20] proved the following theorems for contraction mapping and weakly contractive mapping of integral type, respectively which are generalization of the Banach fixed point theorem.

Theorem 1.2 [8] Let T be a mapping from complete metric space (X, d) into itself satisfying

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq k \int_0^{d(x,y)} \varphi(t)dt \; \forall x, y \in X, \tag{1.3}$$

where $k \in [0, 1)$ is a constant and $\varphi : R^+ \to R^+$ be a Lebesgue-integrable mapping which is summable, nonnegative, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$. Then T has a unique fixed point $z \in X$, such that for each $x \in X$, $lim_{n\to\infty} T^n x = z$.

Theorem 1.3 [20] Let T be a mapping from complete metric space (X, d) into itself satisfying

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq k \int_0^{m(x,y)} \varphi(t)dt \; \forall x, y \in X, \tag{1.4}$$

where $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$, and

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq k \int_0^{M(x,y)} \varphi(t)dt \forall x, y \in X, \tag{1.5}$$

with $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$,
where $k \in [0, 1)$ and $\varphi : R^+ \to R^+$ in both cases is as defined in Theorem 1.1. Then T has a unique fixed point $z \in X$, such that for each $x \in X$, $lim_{n\to\infty} T^n x = z$.

Afterward, many authors extended this work to more general contractive conditions. The works noted in [21, 3, 1, 11, 17].
The following definition is taken from Breind [6]

Definition 1.1 A single valued mapping f on a metric space X is called a weak contraction or $(\delta, L)-$weak contraction if and only if there exists two constants $L \geq 0$ and $\delta \in [0, 1)$ such that

$$d(fx, fy) \leq \delta d(x, y) + L(d(y, fx), \; \forall x, y \in X. \tag{1.6}$$

We shall employ the following definitions in the sequal to obtain our results.

Definition 1.2 [16] A function $\Psi : R^+ \to R^+$ is called a comparison function if it satisfies the following conditions
(i) Ψ is monotone increasing, $\Psi(t) < t$ for some $t > 0$,
(ii) $\Psi(0) = 0$,
(iii) $\lim_{n\to\infty} \Psi^n(t) = 0, \; \forall t \geq 0$.

Definition 1.3 [14, 4] The function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if and only if the following properties are satisfied
1. ψ is continuous and non-decreasing.
2. $\psi(t) = 0 \Leftrightarrow t = 0$.

Afterwards, the outhers in [16], [15] and others continued the study of the existence of fixed points and common fixed points for several contractive mappings of integral type in complete metric spaces. In 2010 M. O. Olatinwo [16] generalized the result of Branciari and established the following fixed point theorems

Theorem 1.4 [16] Let (X, d) be a complete metric space and $f : X \to X$ satisfies a (L, ψ) -weak contraction of integral type

$$\int_0^{d(fx,fy)} \varphi(t)dv(t) \leq L(\int_0^{d(x,fx)} \varphi(t)dv(t))^r(\int_0^{d(y,fx)} \varphi(t)dv(t)) + \psi(\int_0^{d(x,y)} \varphi(t)dv(t))\ \forall x, y \in X, \tag{1.7}$$

where $L \geq 0, r \geq 0$. Suppose that
(i) $\psi : R^+ \to R^+$ is a continuous comparison function and $v : R^+ \to R^+$ is a monotone increasing functions,
(ii) $\varphi : R^+ \to R^+$ be a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dv(t) > 0$.
Then f has a unique fixed point $x^* \in X$ such that for each $x \in X$, $lim_{n\to\infty} f^n x = x^*$.

Theorem 1.5 [16] Let (X, d) be a complete metric space and $f : X \to X$ satisfies a (ϕ, ψ) -weak contraction of integral type

$$\int_0^{d(fx,fy)} \varphi(t)dv(t) \leq \phi(\int_0^{d(x,fx)} \varphi(t)dv(t))(\int_0^{d(y,fx)} \varphi(t)dv(t)) + \psi(\int_0^{d(x,y)} \varphi(t)dv(t))\ \forall x, y \in X. \tag{1.8}$$

Suppose that
(i) $\psi : R^+ \to R^+$ is a continuous comparison function,
(ii) $\Phi, v : R^+ \to R^+$ are monotone increasing functions such that Φ is continuous and $\Phi(0) = 0$,
(iii) $\varphi : R^+ \to R^+$ be a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dv(t) > 0$ and $v : R^+ \to R^+$ is also an increasing function. Then f has a unique fixed point $x^* \in X$ such that for each $x \in X$, $lim_{n\to\infty} f^n x = x^*$.

In 2012, H. Aydi [4] presented the following definition and fixed point theorem for contractive condition of integral type involving altering distances as the following

Definition 1.4 [4] $u : [0, +\infty) \to [0, +\infty)$ is subadditive on each $[a, b] \subset [0, +\infty)$ if,

$$\int_0^{a+b} \varphi(t)dt \leq \int_0^{a} \varphi(t)dt + \int_0^{b} \varphi(t)dt.$$

Theorem 1.6 [4] Let (X, d) be a complete metric space and $f : X \to X$ such that

$$\psi(\int_0^{d(fx,fy))} u(t)dt) \leq \psi(\theta(x, y)) - \Phi(\theta(x, y)), \tag{1.9}$$

for each $x, y \in X$ with non$-$negative real numbers α, β, γ such that
$2\alpha + \beta + 2\gamma < 1$, where ψ, Φ are altering distances, and

$$\theta(x, y) = \alpha \int_0^{d(x,fx)+d(y,fy)} u(t)dt$$

$$+\beta \int_0^{d(x,y)} u(t)dt + \gamma \int_0^{\max\{d(x,fy),d(y,fx)\}} u(t)dt, \tag{1.10}$$

where $u(t) : [0, +\infty) \to [0, +\infty)$ be a Lebesgue$-$integrable mapping which is summable, subadditive on each subset of R^+, non-negative such that for each

$$\epsilon > 0,\ \int_0^{\epsilon} u(t)dt > 0.$$

Then f has a unique fixed point in X.

The notion of modular space, as a generalization of a metric space, was introduced by Nakano[13] in 1950 and redefined and generalized by Musielak and Orlicz[12] in 1959. In the existence of fixed point theory and Banach contraction principle occupies a prominent place in the study of metric spaces, it become a most popular tool in solving problems in mathematical analysis and construct methods in mathematics to solve problems in applied mathematics and sciences. In this article we study and prove the existence of fixed point theorems for mappings in modular spaces.

2 Preliminaries

We will start with a brief recollection of basic concepts and facts in modular spaces and modular metric spaces (see [9], [10], [18], [2] and [7]).

Definition 2.1 [18] Let X be an arbitrary vector space over $K = (R$ or $C)$.
a) A functional $\rho : X \to [0, \infty]$ is called modular if:
(i) $\rho(x) = 0$ iff $x = 0$.
(ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, for all $x \in X$.
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, for all $x, y \in X$.
If (iii) is replaced by:
(iv) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, for all $x, y \in X$, then the modular is called convex modular.
b) If ρ a modular in X, then the set $X_\rho = \{x \in X : \rho(\alpha x) \to 0$ as $\alpha \to 0\}$ is called a modular space.

Remark 2.1 [18] Note that ρ is an increasing function as the following, suppose $0 < a < b$, then, property (iii) with $y = 0$ shows that

$$\rho(ax) = \rho(\frac{a}{b}(bx)) \leq \rho(bx).$$

Definition 2.2 [18] Let X_ρ be a modular space.
a) A sequence $(x_n)_{n \in N}$ in X_ρ is said to be:
(i)$\rho-$convergent to x if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(ii) $\rho-$Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
b) X_ρ is $\rho-$complete if every $\rho-$Cauchy sequence is $\rho-$convergent.
c) A subset $B \subset X_\rho$ is said to be $\rho-$closed if for any sequence
$(x_n)_{n \in N} \subset B$ and $x_n \to x$ we have $x \in B$.
d) A subset $B \subset X_\rho$ is called $\rho-$bounded if $\delta_\rho(B) = sup\ \rho(x - y) < \infty$
for all $x, y \in B$, where $\delta_\rho(B)$ is called the $\rho-$diameter of B.
e)ρ has the Fatou property if:

$$\rho(x - y) \leq lim\ inf\ \rho(x_n - y_n),$$

whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$.
f) ρ is said to satisfy the Δ_2-condition if $\rho(2x_n) \to 0$, whenever $(x_n) \to 0$ as $n \to \infty$.

Remark 2.2 [7] Note that since ρ does not satisfy a priori the triangle inequality, we cannot expect that if $\{x_n\}$ and $\{y_n\}$ are $\rho-$convergent, respectively, to x and y then $\{x_n + y_n\}$ is $\rho-$convergent to $\{x + y\}$, neither that a $\rho-$convergent sequence is$\rho-$Cauchy.

Definition 2.3 [18] Let X_ρ be a modular space, where ρ satisfies the Δ_2-condition. Two self$-$mappings T and h of X are called $\rho-$compatible if $\rho(Thx_n - hTx_n) \to 0$, whenever $(x_n)_{n\in N}$ is a sequence in X_ρ such that $hx_n \to z$ and $Tx_n \to z$ for some point $z \in X_\rho$.

3 Main Results

In this section, we study the existence of a common fixed point for $\rho-$compatible mappings satisfying a $(\phi - \Psi)-$weak contraction of integral type in modular spaces

Theorem 3.1 Let X_ρ be a $\rho-$complete modular space, where ρ satisfies the Δ_2-condition. Suppose $c, l \in R^+$, $c > l$ and $T, h : X_\rho \to X_\rho$ are two $\rho-$compatible mappings such that $T(X_\rho) \subseteq h(X_\rho)$ and

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t)d\nu(t) \leq \Psi(\int_0^{\rho(l(hx-hy))} \varphi(t)d\nu(t))$$

$$+\phi(\int_0^{\rho(l(hx-Tx))} \varphi(t)d\nu(t))(\int_0^{\rho(l(hy-Tx))} \varphi(t)d\nu(t)), \tag{3.1}$$

for each $x, y \in X_\rho$, $\Psi : R^+ \to R^+$ is a continuous compasion function and $\nu, \phi : R^+ \to R^+$ are montone increasing functions, such that $\phi(0) = 0$. Let $\varphi : R^+ \to R^+$ be a Lebesgue$-$Stieltjes integrable mapping which is summable, nonnegative, and such that for each

$$\epsilon > 0, \int_0^{\epsilon} \varphi(t)d\nu(t) > 0. \tag{3.2}$$

If one of h or T is continuous, then there exists a unique common fixed point of h and T.

Proof: Let $\alpha \in R^+$ be the conjugate of $\frac{c}{l}$ such that $\frac{l}{c} + \frac{1}{\alpha} = 1$. Let $x_\circ$ be an arbitrary point of X_ρ and generate inductively the sequence $(Tx_n)_{n\in N}$ as $Tx_n = hx_{n+1}$. For each integer $n \geq 1$, inequality (3.1) shows that

$$\int_0^{\rho(c(Tx_{n-1}-Tx_n))} \varphi(t)d\nu(t) \leq \Psi(\int_0^{\rho(l(hx_{n-1}-hx_n))} \varphi(t)d\nu(t))$$

$$+\phi(\int_0^{\rho(l(hx_{n-1}-Tx_{n-1}))} \varphi(t)d\nu(t))(\int_0^{\rho(l(hx_n-Tx_{n-1}))} \varphi(t)d\nu(t))$$

$$\leq \Psi(\int_0^{\rho(c(Tx_{n-2}-Tx_{n-1}))} \varphi(t)d\nu(t)) + \phi(\int_0^{\rho(c(Tx_{n-2}-Tx_{n-1}))} \varphi(t)d\nu(t))(\int_0^{\rho(c(Tx_{n-1}-Tx_{n-1}))} \varphi(t)d\nu(t))$$

$$\leq \Psi(\int_0^{\rho(c(Tx_{n-2}-Tx_{n-1}))} \varphi(t)d\nu(t)) \leq .... \leq \Psi^n(\int_0^{\rho(c(x-Tx))} \varphi(t)d\nu(t)). \tag{3.3}$$

Taking the limit in (3.3) as $n \to \infty$ yields

$$\lim_{n\to\infty} \int_0^{\rho(c(Tx_{n-1}-Tx_n))} \varphi(t)d\nu(t) = 0, \tag{3.4}$$

since $\lim_{n\to\infty} \Psi^n(\int_0^{\rho(c(x-Tx))} \varphi(t)d\nu(t)) = 0$, using (3.2) and (3.4), we get

$$\lim_{n\to\infty} \rho(c(Tx_{n-1} - Tx_n)) = 0. \tag{3.5}$$

Now we show that $(Tx_n)_{n\in N}$ is $\rho-$Cauchy. If not, then there exists an $\epsilon > 0$ and two sequences of integers $\{n(s)\}, \{m(s)\}$, with $n(s) > m(s) \geq s$, such that

$$\rho(l(Tx_{n(s)} - Tx_{m(s)})) \geq \epsilon \text{ for } s = 1, 2, ... \tag{3.6}$$

We can assume that

$$\rho(l(Tx_{n(s)-1} - Tx_{m(s)})) < \epsilon. \tag{3.7}$$

To prove inequality (3.7) we can see [18].
Again from (3.1), we get

$$\begin{aligned} \int_0^{\rho(c(Tx_{m(s)}-Tx_{n(s)}))} \varphi(t)d\nu(t) &\leq \Psi(\int_0^{\rho(l(hx_{m(s)}-hx_{n(s)}))} \varphi(t)d\nu(t)) \\ &+\phi(\int_0^{\rho(l(hx_{m(s)}-Tx_{m(s)}))} \varphi(t)d\nu(t))(\int_0^{\rho(l(hx_{n(s)}-Tx_{m(s)}))} \varphi(t)d\nu(t)) \end{aligned}$$

$$= \Psi(\int_0^{\rho(l(Tx_{m(s)-1}-Tx_{n(s)-1}))} \varphi(t)d\nu(t))+$$

$$\phi(\int_0^{\rho(l(Tx_{m(s)-1}-Tx_{m(s)}))} \varphi(t)d\nu(t))(\int_0^{\rho(l(Tx_{n(s)-1}-Tx_{m(s)}))} \varphi(t)d\nu(t)), \tag{3.8}$$

moreover,

$$\rho(l(Tx_{m(s)-1} - Tx_{n(s)-1})) = \rho(\frac{\alpha l}{\alpha}(Tx_{m(s)-1} - Tx_{m(s)}) + \rho(\frac{lc}{c}(Tx_{m(s)} - Tx_{n(s)-1})$$

$$\leq \rho(\alpha l(Tx_{m(s)-1} - Tx_{m(s)})) + \rho(c(Tx_{m(s)} - Tx_{n(s)-1})), \tag{3.9}$$

taking the limit as $s \to \infty$ in(3.9) and using (3.5), (3.7) and using Δ_2-condition, we have

$$\int_0^{\rho(l(Tx_{m(s)-1}-Tx_{n(s)-1}))} \varphi(t)d\nu(t)) \leq \int_0^{\epsilon} \varphi(t)d\nu(t)), \tag{3.10}$$

in (3.8), taking the limit as $s \to \infty$ and using (3.10), (3.7) and (3.6), we get

$$\begin{aligned} \int_0^{\epsilon} \varphi(t)d\nu(t)) &\leq \int_0^{\rho(c(Tx_{n(s)}-Tx_{m(s)}))} \varphi(t)d\nu(t)) \leq \Psi(\int_0^{\rho(l(Tx_{n(s)-1}-Tx_{m(s)-1}))} \varphi(t)d\nu(t)) \\ &\leq \Psi(\int_0^{\epsilon} \varphi(t)d\nu(t)) < \int_0^{\epsilon} \varphi(t)d\nu(t)), \end{aligned}$$

which is a contradiction. Therefore, by Δ_2-condition $(Tx_n)_{n\in N}$ is $\rho-$Cauchy.

Since X_ρ is $\rho-$complete, then there exists $z \in X_\rho$ such that

$$\rho(c(Tx_n - z)) \to 0 \text{ as } n \to \infty.$$

If T is continuous, then $T^2x_n \to Tz$ and $Thx_n \to Tz$. Since $\rho(c(hTx_n - Thx_n)) \to 0$, then by $\rho-$compatibility, $hTx_n \to Tz$. We now prove that z is a fixed point of T, if not, we have from (3.1)

$$\int_0^{\rho(c(T^2x_n - Tx_n))} \varphi(t)d\nu(t) \leq \Psi(\int_0^{\rho(l(hTx_n - hx_n))} \varphi(t)d\nu(t))$$

$$+\phi(\int_0^{\rho(l(hTx_n - T^2x_n))} \varphi(t)d\nu(t))(\int_0^{\rho(l(hx_n - T^2x_n))} \varphi(t)d\nu(t)).$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned}\int_0^{\rho(c(Tz-z))} \varphi(t)d\nu(t) &\leq \Psi(\int_0^{\rho(l(Tz-z))} \varphi(t)d\nu(t)) \\ &< \int_0^{\rho(l(Tz-z))} \varphi(t)d\nu(t) < \int_0^{\rho(c(Tz-z))} \varphi(t)d\nu(t).\end{aligned}$$

Leading to a contradiction again. Therefore $z = Tz$. Moreover, $T(X_\rho) \subseteq h(X_\rho)$ and thus, there exists a point $z_1 \in X_\rho$ such that

$$z = Tz = hz_1. \tag{3.11}$$

From (3.1), we obtain

$$\int_0^{\rho(c(T^2x_n - Tz_1))} \varphi(t)d\nu(t) \leq \Psi(\int_0^{\rho(l(hTx_n - hz_1))} \varphi(t)d\nu(t))$$

$$+\phi(\int_0^{\rho(l(hTx_n - T^2x_n))} \varphi(t)d\nu(t))(\int_0^{\rho(l(hz_1 - T^2x_n))} \varphi(t)d\nu(t)),$$

as $n \to \infty$, and using (3.11), we get

$$\int_0^{\rho(c(z-Tz_1))} \varphi(t)d\nu(t) \leq \Psi(0) = 0, \tag{3.12}$$

so that (3.12), we have a contradiction. Therefore by properties of φ, we get $\int_0^{\rho(c(z-Tz_1))} \varphi(t)d\nu(t) = 0$, from which is follows that

$$\rho(c(z - Tz_1)) = 0, \text{ or } z = Tz_1 = hz_1.$$

Also, $hz = hTz_1 = Thz_1 = Tz = z$ (see [18]). Therefore z is a common fixed point of T and h. In addition, if one consider h to be continuous instead of T, then by similar argument as above, one can prove that $Tz = hz = z$.
For uniqueness, suppose that $(w \neq z)$ be another common fixed point of T and h then from (3.1), we get

$$\int_0^{\rho(c(z-w))} \varphi(t)d\nu(t) = \int_0^{\rho(c(Tz-Tw))} \varphi(t)d\nu(t)$$

$$\leq \Psi(\int_0^{\rho(l(hz-hw))} \varphi(t)d\nu(t)) + \phi(\int_0^{\rho(l(hz-Tz))} \varphi(t)d\nu(t))(\int_0^{\rho(l(hw-Tz))} \varphi(t)d\nu(t)),$$

$$\leq \Psi(\int_0^{\rho(l(z-w))} \varphi(t)d\nu(t)) \leq \Psi(\int_0^{\rho(c(z-w))} \varphi(t)d\nu(t)) < (\int_0^{\rho(c(z-w))} \varphi(t)d\nu(t)).$$

Leading to a contradiction again. Therefore, by the condition on φ, we get $\int_0^{\rho(c(z-w))} \varphi(t)d\nu(t) = 0$, from which it follows that

$$\rho(c(z-w)) = 0 \text{ or } z = w.$$

Hence T and h have a unique common fixed point.

The following theorem is another version of Theorem 3.1 when $l = c$, by adding the restrictions that $T, h : B \to B$, where B is a $\rho-$closed and $\rho-$bounded subset of X_ρ.

Theorem 3.2 Let X_ρ be a $\rho-$complete modular space, where ρ satisfies the Δ_2-condition and B is a $\rho-$closed and $\rho-$bounded subset of X_ρ. Suppose $T, h : B \to B$, are $\rho-$compatible mappings such that $T(X_\rho) \subseteq h(X_\rho)$ and

$$\int_0^{\rho(c(Tx-Ty)} \varphi(t)d\nu(t) \leq \Psi(\int_0^{\rho(c(hx-hy))} \varphi(t)d\nu(t))$$

$$+\phi(\int_0^{\rho(c(hx-Tx))} \varphi(t)d\nu(t))(\int_0^{\rho(c(hy-Tx))} \varphi(t)d\nu(t))$$

for each $x, y \in X_\rho$, $\Psi : R^+ \to R^+$ is a continuous compasion function and $\nu, \phi : R^+ \to R^+$ are montone increasing functions, such that $\phi(0) = 0$. Let $\varphi : R^+ \to R^+$ be a Lebesgue$-$Stieltjes integrable mapping which is summable, nonnegative, and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)d\nu(t) > 0.$$

If one of h or T is continuous, then there exists a unique common fixed point of h and T.

Proof: let $x \in B$, $m, n \in N$. Then,

$$\int_0^{\rho(c(Tx_{n+m}-Tx_m)} \varphi(t)d\nu(t) \leq \Psi(\int_0^{\rho(c(hx_{n+m}-hx_m))} \varphi(t)d\nu(t))$$

$$+\phi(\int_0^{\rho(c(hx_{n+m}-Tx_{n+m}))} \varphi(t)d\nu(t))(\int_0^{\rho(c(hx_m-Tx_{n+m}))} \varphi(t)d\nu(t))$$

$$\leq \Psi(\int_0^{\rho(c(Tx_{n+m-1}-Tx_{m-1}))} \varphi(t)d\nu(t)) + \phi(\int_0^{\rho(c(Tx_{n+m-1}-Tx_{n+m}))} \varphi(t)d\nu(t))$$

$$(\int_0^{\rho(c(Tx_{m-1}-Tx_{n+m}))} \varphi(t)d\nu(t)),$$

Taking the limit as $n, m \to \infty$ and using (3.5), we have

$$\lim_{n,m\to\infty} \int_0^{\rho(c(Tx_{n+m}-Tx_m))} \varphi(t)d\nu(t) \leq \lim_{n,m\to\infty} \Psi(\int_0^{\rho(c(Tx_{n+m-1}-Tx_{m-1}))} \varphi(t)d\nu(t))$$

$$\leq \lim_{m\to\infty} \Psi^m(\int_0^{\rho(c(Tx_n-x))} \varphi(t)d\nu(t)) \leq \lim_{m\to\infty} \Psi^m(\int_0^{\delta_\rho(B)} \varphi(t)d\nu(t)).$$

Since B is $\rho-$bounded, then,

$$\lim_{n,m\to\infty} \int_0^{\rho(c(Tx_{n+m}-Tx_m))} \varphi(t)d\nu(t) \leq 0,$$

which implies that $\lim_{n,m\to\infty} \rho(c(Tx_{n+m} - Tx_m) = 0$.
Therefore by Δ_2-condition, $(Tx_n)_{n\in N}$ is $\rho-$Cauchy. Since X_ρ is $\rho-$complete, then there exists $z \in X_\rho$ such that

$$\rho(c(Tx_n - z)) \to 0 \text{ as } n \to \infty.$$

If T is continuous, then $T^2x_n \to Tz$ and $Thx_n \to Tz$. Since $\rho(c(hTx_n - Thx_n)) \to 0$, then by $\rho-$compatibility, $hTx_n \to Tz$.
We now prove that z is a fixed point of T, if not, we have from (3.1)

$$\int_0^{\rho(c(T^2x_n-Tx_n))} \varphi(t)d\nu(t) \leq \Psi(\int_0^{\rho(c(hTx_n-hx_n))} \varphi(t)d\nu(t))$$

$$+\phi(\int_0^{\rho(c(hTx_n-T^2x_n))} \varphi(t)d\nu(t))(\int_0^{\rho(c(hx_n-T^2x_n))} \varphi(t)d\nu(t)).$$

Taking the limit as $n \to \infty$, we get

$$\int_0^{\rho(c(Tz-z))} \varphi(t)d\nu(t) \leq \Psi(\int_0^{\rho(c(Tz-z))} \varphi(t)d\nu(t)) < \int_0^{\rho(c(Tz-z))} \varphi(t)d\nu(t).$$

Leading to a contradiction again. Therefore $z = Tz$. Moreover, $T(X_\rho) \subseteq h(X_\rho)$ and thus there exists a point $z_1 \in X_\rho$ such that

$$z = Tz = hz_1.$$

From (3.1), we obtain

$$\int_0^{\rho(c(T^2x_n-Tz_1))} \varphi(t)d\nu(t) \leq \Psi(\int_0^{\rho(c(hTx_n-hz_1))} \varphi(t)d\nu(t))$$

$$+\phi(\int_0^{\rho(c(hTx_n-T^2x_n))} \varphi(t)d\nu(t))(\int_0^{\rho(c(hz_1-T^2x_n))} \varphi(t)d\nu(t)),$$

as $n \to \infty$, and using (3.11), we get

$$\int_0^{\rho(c(z-Tz_1))} \varphi(t)d\nu(t) \leq \Psi(0) = 0,$$

so that (3.12), we have a contradiction. Therefore by the condition on φ, we get $\int_0^{\rho(c(z-Tz_1))} \varphi(t)d\nu(t) = 0$, from which is follows that

$$\rho(c(z - Tz_1)) = 0, \text{ or } z = Tz_1 = hz_1.$$

Also $hz = hTz_1 = Thz_1 = Tz = z$. Therefore z is a common fixed point of T and h. In addition, if one considers h to be contiuous instead of T, then by similar argument as above, one can prove $Tz = hz = z$.
For uniqueness, suppose that $(z \neq w)$ are two arbitrary common fixed point of T and h, then from (3.1), we get

$$\int_0^{\rho(c(z-w))} \varphi(t)d\nu(t) = \int_0^{\rho(c(Tz-Tw))} \varphi(t)d\nu(t)$$

$$\leq \Psi(\int_0^{\rho(l(hz-hw))} \varphi(t)d\nu(t)) + \phi(\int_0^{\rho(l(hz-Tz))} \varphi(t)d\nu(t))(\int_0^{\rho(l(hw-Tz))} \varphi(t)d\nu(t)),$$

$$\leq \Psi(\int_0^{\rho(l(z-w))} \varphi(t)d\nu(t)) \leq \Psi(\int_0^{\rho(c(z-w))} \varphi(t)d\nu(t)) < (\int_0^{\rho(c(z-w))} \varphi(t)d\nu(t)).$$

Leading to a contradiction again. Therefore, by the conditions on φ, we get $\int_0^{\rho(c(z-w))} \varphi(t)d\nu(t) = 0$, from which it follows that

$$\rho(c(z - w)) = 0 \text{ or } z = w.$$

Hence T and h have a unique common fixed point.

Now, we study the existence of a common fixed point for $\rho-$compatible mappings in modular spaces involving altering distances of integral type in the following Theorem.

Theorem 3.3 Let X_ρ be a $\rho-$complete modular space, where ρ satisfies the Δ_2-condition. Suppose $c, l \in R^+$, $c > l$ and $T, h : X_\rho \to X_\rho$ are two $\rho-$compatible mappings such that $T(X_\rho) \subseteq h(X_\rho)$ and

$$\psi(\int_0^{\rho(c(Tx-Ty))} u(t)dt) \leq \psi(\theta(x, y)) - \Phi(\theta(x, y)), \tag{3.13}$$

for each $x, y \in X_\rho$ with non$-$negative real numbers ζ, β, γ such that
$2\zeta + \beta + 2\gamma < 1$, where ψ, Φ are altering distances, and

$$\theta(x, y) = \zeta \int_0^{\rho(l(hx-Tx)+l(hy-Ty))} u(t)dt$$

$$+\beta \int_0^{\rho(l(hx-hy))} u(t)dt + \gamma \int_0^{\max\{\rho(l(hx-Ty)),\rho(l(hy-Tx))\}} u(t)dt, \tag{3.14}$$

where $u(t) : R^+ \to R^+$ be a Lebesgue$-$integrable mapping which is summable, subadditive on each subset of R^+, nonnegative such that for each

$$\epsilon > 0, \int_0^{\epsilon} u(t)dt > 0. \tag{3.15}$$

If one of h or T is continuous, then there exists a unique fixed point of h and T

Proof: Let $\alpha \in R^+$ be the conjugate of $\frac{c}{l}$ such that $\frac{l}{c} + \frac{1}{\alpha} = 1$. Let x be an arbitrary point of X_ρ and generate inductively the sequence $(Tx_n)_{n \in N}$ as $Tx_n = hx_{n+1}$. For each integer $n \geq 1$, by (3.14), we have for each integer $n \geq 1$,

$$\theta(x_{n+1},x_n)=\zeta\int_0^{\rho(l(hx_{n+1}-Tx_{n+1})+l(hx_n-Tx_n))}u(t)dt$$
$$+\beta\int_0^{\rho(l(hx_{n+1}-hx_n))}u(t)dt+\gamma\int_0^{\max\{\rho(l(hx_{n+1}-Tx_n)),\rho(l(hx_n-Tx_{n+1}))\}}u(t)dt$$
$$=\zeta\int_0^{\rho(l(Tx_n-Tx_{n+1})+l(Tx_{n-1}-Tx_n))}u(t)dt$$
$$+\beta\int_0^{\rho(l(Tx_n-Tx_{n-1}))}u(t)dt+\gamma\int_0^{\max\{\rho(l(Tx_n-Tx_n)),\rho(l(Tx_{n-1}-Tx_{n+1}))\}}u(t)dt, \tag{3.16}$$

by subadditive of u, we have

$$\theta(x_{n+1},x_n)=\zeta\int_0^{\rho(l(Tx_n-Tx_{n+1}))}u(t)dt+\zeta\int_0^{\rho(l(Tx_{n-1}-Tx_n)}u(t)dt$$
$$+\beta\int_0^{\rho(l(Tx_n-Tx_{n-1}))}u(t)dt+\gamma\int_0^{\rho l(Tx_{n-1}-Tx_{n+1})}u(t)dt.$$

Moreover, $(c>l)$ and

$$\rho(l(Tx_{n-1}-Tx_{n+1}))\le\rho(\frac{\alpha l}{\alpha}(Tx_{n-1}-Tx_n))+\rho(\frac{cl}{c}(Tx_n-Tx_{n+1}))$$
$$\le\rho(\alpha l(Tx_{n-1}-Tx_n))+\rho(c(Tx_n-Tx_{n+1})),$$

which implies

$$\theta(x_{n+1},x_n)\le\zeta\int_0^{\rho(c(Tx_n-Tx_{n+1}))}u(t)dt$$
$$+\zeta\int_0^{\rho(c(Tx_{n-1}-Tx_n))}u(t)dt+\beta\int_0^{\rho(c(Tx_n-Tx_{n-1}))}u(t)dt$$
$$+\gamma\int_0^{\rho(c(Tx_{n-1}-Tx_n))}u(t)dt+\gamma\int_0^{\rho(c(Tx_n-Tx_{n+1}))}u(t)dt. \tag{3.17}$$

From (3.13) and (3.17), we have

$$\begin{aligned}\psi(\int_0^{\rho(c(Tx_{n+1}-Tx_n))}u(t)dt) &\le \psi(\theta(x_{n+1},x_n))-\Phi(\theta(x_{n+1},x_n))\le\psi(\theta(x_{n+1},x_n))\\ &= \psi(\zeta\int_0^{\rho(c(Tx_n-Tx_{n+1}))}u(t)dt+\zeta\int_0^{\rho(c(Tx_{n-1}-Tx_n))}u(t)dt\\ &\quad +\beta\int_0^{\rho(c(Tx_n-Tx_{n-1}))}u(t)dt+\gamma\int_0^{\rho(c(Tx_{n-1}-Tx_n))}u(t)dt\\ &\quad +\gamma\int_0^{\rho c(Tx_n-Tx_{n+1}))}u(t)dt).\end{aligned}$$

By the fact ψ is non-decreasing, we get

$$\int_0^{\rho(c(Tx_{n+1}-Tx_n))} u(t)dt \leq (\theta(x_{n+1},x_n)) \leq (\zeta+\gamma) \int_0^{\rho(c(Tx_{n+1}-Tx_n))} u(t)dt$$

$$+(\zeta+\beta+\gamma) \int_0^{\rho(c(Tx_n-Tx_{n-1}))} u(t)dt,$$

which implies that

$$\int_0^{\rho(c(Tx_{n+1}-Tx_n))} u(t)dt \leq \frac{\zeta+\beta+\gamma}{1-\zeta-\gamma} \int_0^{\rho(c(Tx_n-Tx_{n-1}))} u(t)dt.$$

Letting $h = \frac{\zeta+\beta+\gamma}{1-\zeta-\gamma}$. By hypotheses on ζ, β and γ, we get $h \in [0,1)$. By induction, we have

$$\int_0^{\rho(c(Tx_{n+1}-Tx_n))} u(t)dt \leq h \int_0^{\rho(c(Tx_n-Tx_{n-1}))} u(t)dt \leq h^n \int_0^{\rho(c(Tx-x))} u(t)dt.$$

Taking the limit as $n \to \infty$ yields,

$$\lim_n \int_0^{\rho(c(Tx_{n+1}-Tx_n))} u(t)dt \leq 0,$$

thus inequality (3.15) implies that

$$\lim_{n\to\infty} \rho(c(Tx_{n+1}-Tx_n)) \to 0. \tag{3.18}$$

Now, we show that $(Tx_n)_{n\in N}$ is $\rho-$Cauchy. If not, then, there exists an $\epsilon > 0$ and two sequences of integers $\{n(s)\}, \{m(s)\}$, with $n(s) > m(s) \geq s$, such that

$$\rho(l(Tx_{n(s)} - Tx_{m(s)})) \geq \epsilon \text{ for } s = 1, 2, ... \tag{3.19}$$

We can assume that

$$\rho(l(Tx_{n(s)-1} - Tx_{m(s)})) < \epsilon. \tag{3.20}$$

Again from (3.14), we have

$$\theta(x_{m(s)}, x_{n(s)}) = \zeta \int_0^{\rho(l(hx_{m(s)}-Tx_{m(s)})+l(hx_{n(s)}-Tx_{n(s)}))} u(t)dt$$

$$+\beta \int_0^{\rho(l(hx_{m(s)}-hx_{n(s)}))} u(t)dt + \gamma \int_0^{\max\{\rho(l(hx_{m(s)}-Tx_{n(s)})),\rho(l(hx_{n(s)}-Tx_{m(s)}))\}} u(t)dt$$

$$= \zeta \int_0^{\rho(l(Tx_{m(s)-1}-Tx_{m(s)})+l(Tx_{n(s)-1}-Tx_{n(s)}))} u(t)dt + \beta \int_0^{\rho(l(Tx_{m(s)-1}-Tx_{n(s)-1}))} u(t)dt$$

$$+\gamma \int_0^{\max\{\rho(l(Tx_{m(s)-1}-Tx_{n(s)})),\rho(l(Tx_{n(s)-1}-Tx_{m(s)}))\}} u(t)dt, \tag{3.21}$$

moreover,

$$\begin{aligned}\rho(l(Tx_{m(s)-1}-Tx_{n(s)-1})) &\leq \rho(\frac{\alpha l}{\alpha}(Tx_{m(s)-1}-Tx_{m(s)}))+\rho(\frac{cl}{c}(Tx_{m(s)}-Tx_{n(s)-1}))\\ &\leq \rho(\alpha l(Tx_{m(s)-1}-Tx_{m(s)}))+\rho(c(Tx_{m(s)}-Tx_{n(s)-1})),\end{aligned}$$

using the Δ_2-condition and (18), we get

$$\lim_{s\to\infty}\rho(\alpha l(Tx_{m(s)-1}-Tx_{m(s)}))=0. \tag{3.22}$$

Therefore

$$\lim_{s\to\infty}\int_0^{\rho(l(Tx_{m(s)-1}-Tx_{n(s)-1}))}u(t)dt\leq\int_0^{\epsilon}u(t)dt, \tag{3.23}$$

also

$$\begin{aligned}\rho(l(Tx_{m(s)-1}-Tx_{n(s)})) &\leq \rho(\frac{\alpha l}{\alpha}(Tx_{m(s)-1}-Tx_{m(s)}))+\rho(\frac{cl}{c}(Tx_{m(s)}-Tx_{n(s)}))\\ &\leq \rho(\alpha l(Tx_{m(s)-1}-Tx_{m(s)}))+\rho(c(Tx_{m(s)}-Tx_{n(s)})),\end{aligned}$$

using the Δ_2-condition and (3.19), (3.20), (3.22), we have

$$\lim_{s\to\infty}\int_0^{\max\{\rho(l(Tx_{m(s)-1}-Tx_{n(s)})),\rho(l(Tx_{n(s)-1}-Tx_{m(s)}))\}}u(t)dt\leq\int_0^{\epsilon}u(t)dt. \tag{3.24}$$

Taking the limit as $s\to\infty$ in (3.21), using (3.18), (3.23) and (3.24), we have

$$\lim_{s\to\infty}\theta(x_{m(s)},x_{n(s)})\leq(\beta+\gamma)\int_0^{\epsilon}u(t)dt. \tag{3.25}$$

On the other hand, by (3.13)

$$\psi(\int_0^{\rho(c(Tx_{m(s)}-Tx_{n(s)})}u(t)dt)\leq\psi(\theta(x_{m(s)},x_{n(s)}))-\Phi(\theta(x_{m(s)},x_{n(s)}))$$

Taking $s\to\infty$ and using the continuity of ψ and Φ, we have from (3.19), (3.25)

$$\begin{aligned}\psi(\int_0^{\epsilon}u(t)dt)\leq\psi(\int_0^{\rho(c(Tx_{m(s)}-Tx_{n(s)})}u(t)dt)\\ \leq\psi((\beta+\gamma)\int_0^{\epsilon}u(t)dt)-\Phi((\beta+\gamma)\int_0^{\epsilon}u(t)dt)\\ \leq\psi(\int_0^{\epsilon}u(t)dt)-\Phi((\beta+\gamma)\int_0^{\epsilon}u(t)dt),\end{aligned}$$

which implies that $\Phi((\beta+\gamma)\int_0^{\epsilon}u(t)dt)=0$, so by a property of Φ, we get $\int_0^{\epsilon}u(t)dt=0$, that is a contradiction. Thus

$(Tx_n)_{n\in N}$ is $\rho-$Cauchy.
Since X_ρ is $\rho-$complete, then there exists $z \in X_\rho$ such that

$$\rho(c(Tx_n - z)) \to 0 \text{ as } n \to \infty.$$

If T is continuous, then $T^2x_n \to Tz$ and $Thx_n \to Tz$. Since $\rho(c(hTx_n - Thx_n)) \to 0$, then by $\rho-$compatibility, $hTx_n \to Tz$.
We now prove that z is a fixed point of T, we have from (3.14)

$$\theta(Tx_n, x_n) = \zeta \int_0^{\rho(l(hTx_n - T^2x_n) + l(hx_n - Tx_n))} u(t)dt$$
$$+\beta \int_0^{\rho(l(hTx_n - hx_n))} u(t)dt + \gamma \int_0^{\max\{\rho(l(hTx_n - Tx_n)), \rho(l(hx_n - T^2x_n))\}} u(t)dt.$$

Taking the limit as $n \to \infty$, yields

$$\lim_{n\to\infty} \theta(Tx_n, x_n) = (\beta + \gamma) \int_0^{\rho(l(Tz - z))} u(t)dt. \tag{3.26}$$

Again, from (3.13)

$$\psi(\int_0^{\rho(c(T^2x_n - Tx_n))} u(t)dt) \leq \psi(\theta(Tx_n, x_n)) - \Phi(\theta(Tx_n, x_n)),$$

as $n \to \infty$ and using (3.26), we get

$$\psi(\int_0^{\rho(c(Tz-z))} u(t)dt) \leq \psi((\beta + \gamma) \int_0^{\rho(l(Tz-z))} u(t)dt) - \Phi((\beta + \gamma) \int_0^{\rho(l(Tz-z))} u(t)dt),$$

using the continuity of ψ and Φ, we obtain

$$\begin{aligned} \psi(\int_0^{\rho(c(Tz-z))} u(t)dt) &\leq \psi((\beta + \gamma) \int_0^{\rho(c(Tz-z))} u(t)dt) \\ &\quad -\Phi((\beta + \gamma) \int_0^{\rho(c(Tz-z))} u(t)dt) \\ &\leq \psi(\int_0^{\rho(c(Tz-z))} u(t)dt) - \Phi((\beta + \gamma) \int_0^{\rho(c(Tz-z))} u(t)dt), \end{aligned}$$

which implies that $\Phi((\beta + \gamma) \int_0^{\rho(c(Tz-z))} u(t)dt) = 0$, so by properties of Φ and u, we get $\rho(c(Tz - z)) = 0$ or $z = Tz$.
Moreover, $T(X_\rho) \subseteq h(X_\rho)$ and thus there exists a point $z_1 \in X_\rho$ such that

$$z = Tz = hz_1. \tag{3.27}$$

From (3.14), we have

$$\theta(Tx_n, z_1) = \zeta \int_0^{\rho(l(hTx_n - T^2x_n)+l(hz_1-Tz_1))} u(t)dt$$
$$+\beta \int_0^{\rho(l(hTx_n-hz_1))} u(t)dt + \gamma \int_0^{\max\{\rho(l(hTx_n-Tz_1)),\rho(l(hz_1-T^2x_n))\}} u(t)dt,$$

as $n \to \infty$, yields

$$\lim_{n\to\infty} \theta(Tx_n, z_1) = \zeta \int_0^{\rho(l(Tz-Tz)+l(hz_1-Tz_1))} u(t)dt$$
$$+\beta \int_0^{\rho(l(Tz-hz_1))} u(t)dt + \gamma \int_0^{\max\{\rho(l(Tz-Tz_1)),\rho(l(hz_1-Tz))\}} u(t)dt,$$

using (3.27), we get

$$\lim_{n\to\infty} \theta(Tx_n, z_1) = (\zeta+\gamma) \int_0^{\rho(l(z-Tz_1))} u(t)dt. \tag{3.28}$$

Again from (3.13), we have

$$\psi(\int_0^{\rho(c(T^2x_n-Tz_1))} u(t)dt) \leq \psi(\theta(Tx_n, z_1)) - \Phi(\theta(Tx_n, z_1)).$$

Taking the limit as $n \to \infty$ and using (3.27) and (3.28), we get

$$\psi(\int_0^{\rho(c(z-Tz_1))} u(t)dt) \leq \psi((\zeta+\gamma)\int_0^{\rho(l(z-Tz_1))} u(t)dt) - \Phi((\zeta+\gamma)\int_0^{\rho(l(z-Tz_1))} u(t)dt)$$

$$\leq \psi((\zeta+\gamma)\int_0^{\rho(c(z-Tz_1))} u(t)dt) - \Phi((\zeta+\gamma)\int_0^{\rho(c(z-Tz_1))} u(t)dt)$$

$$\leq \psi(\int_0^{\rho(l(z-Tz_1))} u(t)dt) - \Phi((\zeta+\gamma)\int_0^{\rho(l(z-Tz_1))} u(t)dt),$$

which implies that $\Phi((\zeta+\gamma)\int_0^{\rho(l(z-Tz_1))} u(t)dt) = 0$, so by properties of Φ and u ,we get $\rho(c(z-Tz_1)) = 0$, therefore $z = Tz_1 = hz_1$ and also $hz = hTz_1 = Thz_1 = Tz = z$.
In addition, if one consider h to be a continuous in stead of T,
then by similar argument (as above), one can prove

$$hz = Tz = z.$$

Finally, suppose that z and w are two arbitrary common fixed point of T and h, $(w \neq z)$, then from (3.14), we get

$$\theta(z, w) = \zeta \int_0^{\rho(l(hz-Tz)+l(hw-Tw))} u(t)dt$$
$$+\beta \int_0^{\rho(l(hz-hw))} u(t)dt + \gamma \int_0^{\max\{\rho(l(hz-Tw)),\rho(l(hw-Tz))\}} u(t)dt$$

$$= (\beta+\gamma) \int_0^{\rho(l(z-w))} u(t)dt.$$

From (3.13), we have

$$\psi(\int_0^{\rho(c(Tz-Tw))} u(t)dt) \leq \psi(\theta(z,w)) - \Phi(\theta(z,w))$$

$$\begin{aligned}
\psi(\int_0^{\rho(c(z-w))} u(t)dt) &= \psi(\int_0^{\rho(c(Tz-Tw))} u(t)dt) \\
&\leq \psi((\beta+\gamma) \int_0^{\rho(l(z-w))} u(t)dt) - \Phi((\beta+\gamma) \int_0^{\rho(l(z-w))} u(t)dt) \\
&\leq \psi((\beta+\gamma) \int_0^{\rho(c(z-w))} u(t)dt) - \Phi((\beta+\gamma) \int_0^{\rho(c(z-w))} u(t)dt) \\
&\leq \psi(\int_0^{\rho(c(z-w))} u(t)dt) - \Phi((\beta+\gamma) \int_0^{\rho(c(z-w))} u(t)dt),
\end{aligned}$$

also by properties of Φ and u, we get

$$\Phi((\beta+\gamma) \int_0^{\rho(c(z-w))} u(t)dt) = 0 \Longrightarrow \rho(c(z-w)) = 0 \text{ or } z = w.$$

This complete the prove of this Theorem.

If we take $\psi(t) = \frac{t}{2}$ and $\Phi(t) = \frac{t}{4}$ in Theorem 3.3 we get the following Corollary

Corollary 3.1 Let X_ρ be a $\rho-$complete modular space, where ρ satisfies the Δ_2-condition. Suppose $c, l \in R^+$, $c > l$ and $T, h : X_\rho \to X_\rho$ are two $\rho-$compatible mappings such that $T(X_\rho) \subseteq h(X_\rho)$ and

$$\int_0^{\rho(c(Tx-Ty))} u(t)dt \leq \zeta \int_0^{\rho(l(hx-Tx)+l(hy-Ty))} u(t)dt$$
$$+\beta \int_0^{\rho(l(hx-hy))} u(t)dt + \gamma \int_0^{\max\{\rho(l(hx-Ty)),\rho(l(hy-Tx))\}} u(t)dt$$

for each $x, y \in X_\rho$ with non$-$negative real numbers ζ, β, γ such that $2\zeta+\beta+2\gamma < 1$, where ψ, Φ are altering distences, where $u(t) : R^+ \to R^+$ be a Lebesgue$-$integrable mapping which is summable, subadditive on each subset of R^+, nonnegative, and such that for each

$$\epsilon > 0, \int_0^{\epsilon} u(t)dt > 0.$$

If one of h or T is continuous, then there exists a unique common fixed point of h and T.

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